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Transient temperature distributions in a cylindrical superheating fuel element

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TRANSIENT TEMPERATURE DISTRIBUTIONS IN A CYLINDRICAL SUPERHEATING FUEL ELEMENT

by

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INTRODUCTION

In boiling water reactors the temperature of the steam is limited as also is the thermodynamic efficiency. The steam produced is saturated and this requires the use of more expensive superheating equipment externally than for superheated steam internally. One approach to improving the efficiency is to use the reactor itself as a superheater.

For boiling water reactors, the nuclear superheater, consisting of boiling region and superheating region, may be either of the "integral" type or "separate" type. The integral type nuclear superheater produces steam in the same reactor and the separate type utilizes steam from another reactor. The boiling region can be either in the center or at the periphery of the core, with superheating at the periphery or center, respectively.

From reactor safety and ecomonic standpoints, a study of the fuel elements in nuclear superheating power reactors, which are usually operating at high temperatures, has been increasingly important, since the fuel element is the central and significant component in the power reactor systems and the great amount of power from the nuclear power reactor is demanded. Therefore, a precise knowledge of possible temperature distributions is required by those who design various parts of nuclear reactors.

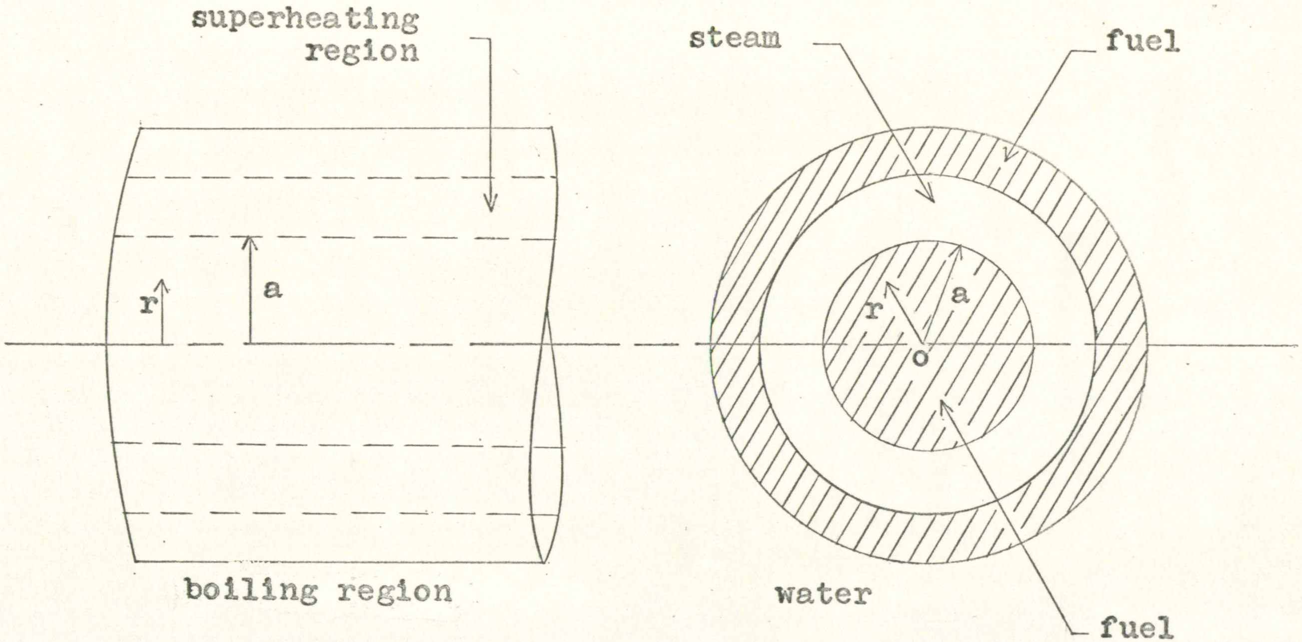

Figure 1. A typical double-annular fuel element of nuclear superheater (integral type)

Analytical methods are available for predicting temperature distributions in reactor fuel elements. One way of predicting the transient temperature distributions in the fuel element is by mathematical analysis. This generally requires an integral solution of a certain form of the general equation for heat conduction in an isotropic solid, i.e., (see Nomenclature)

$$c(x,y,z,T)\rho(x,y,z,T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}k(x,y,z,T)\frac{\partial T}{\partial x} + \frac{\partial}{\partial y}k(x,y,z,T)\frac{\partial T}{\partial y} + \frac{\partial}{\partial z}k(x,y,z,T)\frac{\partial T}{\partial z} + q(x,y,z,t)$$

Temperature, T, is a function of space as well as time.

The solution and its application to transient temperature problems in nuclear superheaters is complicated by the fact that when the reactivity is perturbed, or in many cases, is decaying, the heat generation rate in the fuel elements is a function of time as well as a function of position. Furthermore, the boundary conductance may also change markedly with time.

Considering the many difficulties, the utilization of the technique in this work has been restricted to problems involving a certain geometry. Of the various geometrical shapes in which fuel elements can be fabricated, the solid cylinder is one of the most common. The cylindrical fuel elements in most reactors are relatively long with respect to their radii and it is supposed that the thermal neutron flux distribution is axially symmetric with the fuel element.

In order to simplify the work, while remaining sufficiently general to be applicable, with a minimum of idealizations, to the problems of transient heat conduction in long solid cylindrical reactor fuel elements, the heat generating element is assumed to be a solid cylinder of infinite longitudinal extent. It is also assumed to have axially symmetric temperature and heat-generation-rate distributions. The conducting medium is also assumed to be isotropic and homogeneous.

OBJECTIVE OF INVESTIGATION

The objective of this research is: 1) To examine the transient temperature distribution and heat-transmission-rate distribution in the superheating region in a long solid cylindrical fuel element. The transient temperatures and heat transmission rate are functions of time and radial position. 2) To develop methods of application of the solution to reactor heat transfer problems which involve variable boundary conditions.

REVIEW OF LITERATURE

Several modes of attacking the problems of transient temperature distributions in nuclear reactors has been examined for particular cases to obtain engineering solutions.

Carslaw and Jaeger (1), and Schneider (2) have conducted the most extensive work on the temperature distributions in various geometries and with various boundary conditions.

Ma (3, 4, 5) has examined the stability of cylindrical reactor fuel elements and presented transient temperatures distributions in cylindrical fuel elements.

Loretan (6) has solved the transient multi-dimensional temperature distributions, which is quite general and applies to any geometry, by using a Laplace-variational method.

Tippets (7) estimated the transient temperature distributions in a nuclear fuel element by assuming that boundary conductance and coolant temperature are constant or initially varied.

Snedden (8) has employed the "Dini series associated with a function f(r)" to define the "finite Hankel transform" particularly for use in solving the equation of heat conduction for an infinite solid cylinder, and has outlined (9) a method for solving the transient temperature distribution in an infinite solid cylinder having axially symmetric, time dependent sources of heat.

Kaplan (10) and Hildebrand (11) have presented some techniques for solving the somewhat complicated differential and integral equations which appear in the theoretical analysis of heat transfer problems such as the one considered in this research.

THEORETICAL ANALYSIS

A general form of the differential equation for heat conduction in solids can be written as

$$\frac{\partial}{\partial x} k(x, y, z, T)\frac{\partial T}{\partial x} + \frac{\partial}{\partial y} k(x, y, z, T)\frac{\partial T}{\partial y} + \frac{\partial}{\partial z} k(x, y, z, T)\frac{\partial T}{\partial z}$$

$$+ q(x, y, z, t) = c(x, y, z, T)\rho(x, y, z)\frac{\partial T}{\partial t} \quad (1)$$

In the general case the properties k, c and ρ are functions of the space coordinates and of the temperature, and q, a volumetric heat source, may also be a function of time. Since this is a differential equation, any solution gives rise to arbitrary functions which have to be evaluated in terms of the boundary conditions. However, if k, c, and ρ can be taken as constant, the use of a Laplacian operator makes a considerable simplification possible

$$c\rho\frac{\partial T}{\partial t} = \nabla\cdot(k\nabla T) + q(x, y, z, t) \quad (2a)$$

The fuel element is assumed to be a long solid cylinder having 1) an initially steady state temperature distribution, 2) axial symmetry of temperature and heat generation, 3) constant properties, coolant temperature and boundary conductance and 4) a heat generation rate which varies radially according to the thermal neutron diffusion theory and which is an arbitrary function of time.

If the material of the element is isotropic and homogeneous and its longitudinal heat conduction is negligible, the equation of heat conduction can be expressed as

$$c\rho\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + q(r,t) \tag{2b}$$

By dividing equation 2b by the product $c\rho$, defining $\alpha = \frac{k}{c\rho}$, and assuming that $q(r,t)$ can be written as

$$q(r,t) = q_o\ m(r)\ n(t) \tag{3}$$

where q_o is the initial steady state heat generation rate per unit volume in the fuel element, $m(r)$ and $n(t)$ are functions of radius alone and time alone, respectively, the final equation of heat conduction, which will be used as the basic equation, can be expressed as

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) + \frac{q_o}{c\rho}\, m(r)n(t) \tag{4}$$

The boundary condition assumed on equation 4 will be

$$\frac{\partial T}{\partial r} = 0, \qquad r = 0 \tag{5}$$

Newton's law of cooling (1) gives another boundary condition in the form

$$\frac{\partial T}{\partial r} + hT = 0, \quad r = a \tag{6a}$$

where

$$h = \frac{U}{k} > 0 \tag{6b}$$

Equations 5 and 6a are boundary conditions which will be used to solve the basic equation 4.

As a preparatory step, the finite Hankel transform and its important property will be evaluated in order to solve the basic equation. Equation 4 can be integrated, after Sneddon (9), by using the finite Hankel transformation of zero order for the cylindrical region (0,a), i.e., $0 \leq r \leq a$. The finite Hankel transformation of zero order of f(r) is defined by the linear functional operator

$$\bar{j}_o\{f(r)\} \equiv \int_0^a rf(r)\, J_o(\lambda_m r)\, dr \equiv \bar{f}_o(\lambda_m),\ 0 \leq r \leq a \quad (7)$$

which is valid for all f(r) integrable in the interval $0 \leq r \leq a$. The set of parameters λ_m can be chosen in more than one way, the choice being determined by the form of the inversion theorem used. If the parameters λ_m are roots of the equation,

$$J_o'(\lambda_m a) + hJ_o(\lambda_m a) = 0,\ m = 1,2,3,\ldots \quad (8a)$$

or in another form,

$$hJ_o(\lambda_m a) - \lambda_m J_1(\lambda_m a) = 0,\ m = 1,\ 2,\ 3,\ \ldots \quad (8b)$$

then a corresponding inversion formula may be used to express f(r) in terms of the set of functions $\bar{f}_o(\lambda_m)$:

$$f(r) = \frac{2}{a^2} \sum_m \frac{\lambda_m^2\, \bar{f}_o(\lambda_m)}{h^2 + \lambda_m^2} \cdot \frac{J_o(\lambda_m r)}{J_o(\lambda_m a)^2},\ 0 < h \quad (9)$$

where $$J_o'(\lambda_m a) = -\lambda_m J_1(\lambda_m a)$$

the summation extending over all positive roots of equation 8b.

In order to derive the required porperty of the transformation $\bar{j}_o$, the notation $T' = dT/dr$ and $T'' = d^2T/dr^2$ is adopted. The following is the sequence of integration by parts in the region (0,a), i.e., $0 \leq r \leq a$.

$$\int_0^a r\left(\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr}\right) J_o(\lambda_m r)\, dr$$

$$= \int_0^a rJ_o(\lambda_m r) \frac{d^2T}{dr^2}\, dr + \int_0^a \frac{dT}{dr} J_o(\lambda_m r)\, dr$$

$$= rT' J_o(\lambda_m r)\Big]_0^a - \int_0^a T'\left\{\frac{d}{dr}[rJ_o(\lambda_m r)] - J_o(\lambda_m r)\right\} dr$$

$$= aT'(a)J_o(\lambda_m a) - \int_0^a T'\lambda_m rJ_o'(\lambda_m r)\, dr$$

$$= aT'(a)J_o(\lambda_m a) - [\lambda_m rTJ_o'(\lambda_m r)]_0^a$$

$$+ \lambda_m \int_0^a T[\lambda_m rJ_o''(\lambda_m r) + J_o'(\lambda_m r)]\, dr$$

$$= aT'(a)J_o(\lambda_m a) - a\lambda_m T(a)J_o'(\lambda_m a)$$

$$+ \lambda_m^2 \int_0^a rT[\, J_o''(\lambda_m r) + \frac{1}{\lambda_m r} J_o'(\lambda_m r) + J_o(\lambda_m r)]dr$$

$$- \lambda_m^2 \int_0^a rTJ_o(\lambda_m r)dr \qquad (10)$$

An application of equations 5, 6a, 8b and the first solution of Bessel's equation, i.e.,

$$J_o''(\lambda_m r) + \frac{1}{\lambda_m r} J_o'(\lambda_m r) + J_o(\lambda_m r) = 0 \qquad (11)$$

to equation 10 gives the reduced form of the equation 10

$$\int_0^a r(T'' + \frac{1}{r} T')\, J_o(\lambda_m r)\, dr = -\lambda_m^2 \int_0^a rTJ_o(\lambda_m r)\, dr \quad (12a)$$

If one compares equation 7, the definition of the finite Hankel transform, to equation 12a, the latter can have another form,

$$\bar{J}_o\left[\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr}\right] = -\lambda_m^2\ \bar{J}_o[T] \qquad (12b)$$

which is a very important property to be used in performing the integration of the basic equation 4.

The definition of the finite Hankel transform, equation 7, is used to transform the equation 4 by first multiplying $rJ_o(\lambda_m r)$ on both sides and then integrating the result between $r = 0$ and $r = a$ in order to obtain the form,

$$\int_0^a \frac{\partial T}{\partial t}\ rJ_o(\lambda_m r)dr - \alpha\int_0^a r\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right) J_o(\lambda_m r)dr$$

$$= \frac{q_o}{c\rho}\ n(t)\int_0^a rm(r)J_o(\lambda_m r)dr \qquad (13)$$

If T is continuous, has a continuous derivative $\frac{\partial T}{\partial t}$ in a domain of the rt plane, and has a value which depends on the choice of t, "Leibnitz's rule", as stated by Kaplan (10), may be applied and the first integral of equation 13 can be expressed as

$$\int_0^a \frac{\partial T}{\partial t}\ rJ_o(\lambda_m r)dr = \frac{d}{dt}\int_0^a rTJ_o(\lambda_m r)dr \qquad (14)$$

Equation 14, equation 7, which is the definition of the finite Hankel transform, and equation 12b, which is its property are used to reduce the equation 13 in the form of the ordinary linear homogeneous equation

$$\frac{d\bar{T}}{dt} + \alpha\lambda_m^2\bar{T} = \frac{q_o}{c\rho}\ \bar{m}(r)n(t) \qquad (15)$$

where

$$\bar{T} = \int_0^a rT(r,t)J_o(\lambda_m r)dr \qquad (16)$$

and

$$\bar{m}(r) = \int_0^a rm(r)J_o(\lambda_m r)dr \qquad (17)$$

Integration of equation 15 with respect to t between $t = t_{i-1}$ and $t = t_i$ gives

$$\bar{T}(r,t_i) = \bar{T}(r,t_{i-1})\, e^{-\alpha\lambda_m^2(t_i - t_{i-1})}$$

$$+ \frac{q_o}{c\rho}\bar{m}(r)\, e^{-\alpha\lambda_m^2 t_i}\int_{t_{i-1}}^{t_i} n_i(t)\, e^{\alpha\lambda_m^2 t}\, dt \qquad (18a)$$

where the subscript i in $n_i(t)$ indicates its correspondence to the time increment. By letting $\Delta t_i = t_i - t_{i-1}$, equation 18a can be written as

$$\bar{T}(r,t_i) = \bar{T}(r,t_{i-1})\, e^{-\alpha\lambda_m^2\Delta t_i}$$

$$+ \frac{q_o}{c\rho}\bar{m}(r)\, e^{-\alpha\lambda_m^2 t_i}\int_{t_{i-1}}^{t_i} n_i(t)\, e^{\alpha\lambda_m^2 t}\, dt \qquad (18b)$$

Substitution of equation 18b into the inversion formula, equation 9, gives the boundary value solution of equation 4

$$T(r,t_i) = \frac{2}{a^2}\sum_{m=1}^{\infty}\frac{\lambda_m^2 J_o(\lambda_m r)}{(h^2 + \lambda_m^2)J_o^2(\lambda_m a)}\Big[\bar{T}(r,t_{i-1})e^{-\alpha\lambda_m^2\Delta t_i}$$

$$+ \frac{q_o}{c\rho}\bar{m}(r)\, e^{-\alpha\lambda_m^2 t_i}\int_{t_{i-1}}^{t_i} n_i(t)\, e^{\alpha\lambda_m^2 t}\, dt\Big], \; h > 0 \qquad (19a)$$

or, by applying equation 8b, gives

$$T(r,t_i) = \frac{2}{a^2} \sum_{m=1}^{\infty} \frac{J_o(\lambda_m r)}{(1 + \frac{\lambda_m^2}{h^2}) J_1^2(\lambda_m a)} \left[\bar{T}(r,t_{i-1}) e^{-\alpha\lambda_m^2 \Delta t_i} \right.$$

$$\left. + \frac{q_o}{c\rho} \bar{m}(r)\; e^{-\alpha\lambda_m^2 t_i} \int_{t_{i-1}}^{t_i} n_i(t)\; e^{\alpha\lambda_m^2 t}\; dt \right], \quad h > 0 \qquad (19b)$$

The boundary value solution of equation 4 given by equation 19b is used to develop particular solutions for certain transient heat transfer problems, which are common to nuclear engineering.

The solution which will be developed is applicable to long solid cylindrical fuel elements having uniform temperature-independent material properties, axial symmetry of heat conduction and generation, and being placed in a uniform thermal neutron flux field. It is assumed that the heat generation varies with both time and radial position and the boundary conductance is greater than zero, i.e., $h > 0$.

It is convenient to consider an infinite solid cylindrical fuel element cooled by the transmission of heat from its surface across a thermal resistance to a uniform coolant. It is also convenient to assume that the radial distribution of the thermal neutron flux in the fuel element is in accord with that predicted for a single element by the elementary thermal neutron diffusion theory given by Glasstone and Edlund (12).

Thermal diffusion equation for the fuel element is given by

$$D\nabla^2\phi - \Sigma_a\phi = 0 \qquad (20a)$$

where ϕ is thermal neutron flux at any point in the fuel element, and D and Σ_a are corresponding diffusion coefficient and macroscopic absorption cross section, respectively. There is no thermal neutron source term in equation 20a, since it is assumed that there is no slowing down in the fuel element. Equation 20a is developed in the form,

$$\nabla^2\phi - \kappa^2\phi = 0 \qquad (20b)$$

where $\kappa^2 = \frac{\Sigma_a}{D}$ and κ is the inverse of the thermal diffusion length, assumed to be uniformly constant.

In cylindrical coordinates, equation 20b can be expressed as

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \kappa^2\phi = 0 \qquad (20c)$$

Since κ^2 is positive, equation 20c is equivalent to a modified Bessel equation and a general solution is of the form,

$$\phi = A_1 I_o(\kappa r) + A_2 K_o(\kappa r) \qquad (21)$$

where I_o and K_o are zero order modified Bessel functions of the first and second kinds, respectively. The second term of equation 21 requires that the neutron flux become infinite at the axis of the fuel element, where r = 0; therefore, A_2 must be zero. Thus, thermal neutron flux distribution in a single infinite solid cylindrical fuel element will be

$$\phi = A_1 I_o(\kappa r) \qquad (22)$$

According to the elementary thermal neutron diffusion theory, equation 22 represents the radial distribution of thermal neutron flux in an infinite solid cylindrical fuel element with conditions of axially symmetric incidence of thermal neutron flux on the boundaries of the element.

Relationships for the steady state temperature distribution and radial distribution of heat generation rate per unit volume in an infinite solid cylindrical fuel element will be derived by using equation 22.

As a good approximation, Glasstone and Sesonske (13) state that the heat generation rate per unit volume at a point in the fuel element is linearly proportional to the thermal neutron flux at that point. Thus, the heat generation rate per unit volume at any point, r, in an infinite solid cylinder may be expressed as

$$q(r) = BI_o(\kappa r) \qquad (23)$$

where B is a constant of proportionality.

Under steady state conditions, i.e., $\frac{\partial T}{\partial t} = 0$ and $n(t) = 1$, the equation of heat conduction, equation 4, can be expressed as

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = - \frac{q(r)}{k}$$

$$= - \frac{B}{k} I_o(\kappa r) \qquad (24a)$$

Multiplication of equation 24a by r dr and integration yields

$$r\frac{dT}{dr} = - \frac{B}{k\kappa} rI_1(\kappa r) + C \qquad (24b)$$

where C is the constant of integration. When r = 0, C = 0. Thus, equation 24b becomes

$$r\frac{dT}{dr} = -\frac{B}{k\kappa} rI_1(\kappa r), \quad r = 0 \tag{24c}$$

Multiplication of equation 24c by dr/r and integration between r = r, T = T and r = a, T = T(a) gives

$$T(r) = T(a) + \frac{B}{k\kappa^2}[I_0(\kappa a) - I_0(\kappa r)] \tag{24d}$$

Under steady state conditions, the total heat transfer rate per unit length across the surface of the element, Q_0, is given by the heat balance, using equation 23

$$Q_0 = 2\pi B\int_0^a rI_0(\kappa r)\,dr$$

$$= \frac{2\pi B}{\kappa} aI_1(\kappa a) \tag{25}$$

Equation 25 solved for B yields

$$B = \frac{\kappa Q_0}{2\pi aI_1(\kappa a)} \tag{26}$$

Substitution of equation 26 into equation 24d gives the steady state temperature distribution

$$T(r,0) = T(a,0) + \frac{Q_0}{2\pi ka\kappa I_1(\kappa a)}[I_0(\kappa a) - I_0(\kappa r)] \tag{27}$$

By putting equation 26 into equation 23 results in the steady state radial distribution of heat generation rate per unit volume

$$q(r,0) = \frac{Q_0\kappa}{2\pi aI_1(\kappa a)} I_0(\kappa r) \tag{28}$$

It has been tacitly assumed that the condition of axial symmetry of thermal neutron flux and the magnitude of κ does not change with time under the transient conditions. Hence, the heat generation rate per unit volume may be expressed for the transient case as

$$q(r,t) = \frac{Q_o \kappa}{2\pi a I_1(\kappa a)} I_o(\kappa r) \cdot n(t) \qquad (29)$$

where n(t) is a function of time alone. Equation 29 can be used in case of heat generation rate dependent on the radius of the fuel material.

If the heat generation rate per unit volume in an infinite solid cylinder is assumed to be constant with respect to position, its steady state magnitude can be expressed by the thermal energy balance as

$$q = \frac{Q_o}{\pi a^2} \qquad (30)$$

where Q_o is the steady state heat transfer rate per unit length across the boundary of the fuel material. For the transient condition, if the term expressed as equation 3

$$q = q_o m(r)\, n(t)$$

is adopted, a comparison with equation 30 gives

$$m(r) = 1 \qquad (31)$$

$$q_o = \frac{Q_o}{\pi a^2} \qquad (32)$$

and $$n(t) = 1, \text{ at } t = 0 \qquad (33)$$

By using equation 30, the equation of heat conduction for the steady state, equation 24a can be expressed as

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{Q_o}{\pi k a^2} \qquad (34)$$

A performance of successive integration of equation 34 using the boundary condition $a\left(\frac{dT}{dr}\right)_a = -\frac{Q_o}{2\pi k}$ from a consideration of the steady state energy balance over the fuel material, results in the steady state temperature distribution

$$T(r,0) = T(a,0) + \frac{Q_o}{4\pi k}\left(1 - \frac{r^2}{a^2}\right) \qquad (35)$$

Equation 35 can be used when the heat generation rate is independent of radius of the fuel element.

It is shown that the steady state temperature distribution and radial distribution of heat generation in the fuel material are equations 27 and 28, which will now be used to develop equation 19b.

The temperature, $T(a,0)$, in equation 27, is measured above any arbitrary coolant temperature. In general, if the coolant is maintained at a temperature T_c, then by Newton's law of cooling

$$T(a,0) = T_c + \frac{Q_o}{2\pi a U}$$

$$= T_c + \frac{Q_o}{2\pi a k h} \qquad (36)$$

If the form $q = q_o\, m(r)\, n(t)$ can be adopted, it is seen from equation 29 that

$$q_o = \frac{Q_o \kappa}{2\pi a I_1(\kappa a)} \tag{37}$$

and

$$m(r) = I_o(\kappa r) \tag{38}$$

By the definition of equation 7

$$\bar{m}(r) = \int_0^a r I_o(\kappa r) J_o(\lambda_m r)\, dr$$

$$= \frac{a}{\kappa^2 + \lambda_m^2} [\kappa I_1(\kappa a) J_o(\lambda_m a)$$

$$+ \lambda_m I_o(\kappa a) J_1(\lambda_m a)] \tag{39a}$$

Use of equation 37, the definition, $\alpha = \frac{k}{c\rho}$, and multiplication of equation 39a by $\frac{q_o}{c\rho}$ result in

$$\frac{q_o}{c\rho} \cdot \bar{m}(r) = \frac{\kappa Q_o \alpha}{2\pi k I_1(\kappa a)(\kappa^2 + \lambda_m^2)} \cdot$$

$$\cdot [\kappa I_1(\kappa a) J_o(\lambda_m a) + \lambda_m I_o(\kappa a) J_1(\lambda_m a)] \tag{39b}$$

Finally, by applying equation 8b and collecting terms, equation 39b becomes

$$\frac{q_o}{c\rho} \cdot \bar{m}(r) = \frac{\kappa Q_o \alpha J_1(\lambda_m a) \lambda_m}{2\pi k(\kappa^2 + \lambda_m^2)} \left[\frac{\kappa}{h} + \frac{I_o(\kappa a)}{I_1(\kappa a)}\right] \tag{39c}$$

Equation 27 can be transformed to obtain, after rearrangement,

$$\bar{T}(r,0) = \frac{a J_1(\lambda_m a)}{\lambda_m} \left\{ T(a,0) + \frac{Q_o}{2\pi k a \kappa} \left[\frac{I_o(\kappa a)}{I_1(\kappa a)} \right.\right.$$

$$\left.\left. - \frac{\lambda_m^2}{\kappa^2 + \lambda_m^2} \cdot \left(\frac{\kappa}{h} + \frac{I_o(\kappa a)}{I_1(\kappa a)} \right) \right] \right\} \tag{40}$$

For the time interval, $0 \leq t \leq t_1$, equation 19b can be expressed in the form

$$T_1(r,t_1) = \frac{2}{a^2} \sum_{m=1}^{\infty} \frac{J_0(\lambda_m r) e^{-\alpha \lambda_m^2 t_1}}{(1 + \frac{\lambda_m^2}{h^2}) J_1^2(\lambda_m a)} \Big\{ \bar{T}(r,0) +$$

$$+ \frac{q_0}{c\rho} \bar{m}(r) \int_0^{t_1} n_1(t) e^{\alpha \lambda_m^2 t} dt \Big\} \quad (41)$$

Substitution of equations 39c and 40 into equation 41 results in

$$T_1(r,t_1) = \frac{2}{a^2} \sum_{m=1}^{\infty} \frac{J_0(\lambda_m r) e^{-\alpha \lambda_m^2 t_1}}{(1 + \frac{\lambda_m^2}{h^2}) \lambda_m J_1(\lambda_m a)} \Big\{ T_1(a,0) +$$

$$+ \frac{Q_0}{2\pi k a \kappa} \Big[\frac{I_0(\kappa a)}{I_1(\kappa a)} - \frac{\lambda_m^2}{\kappa^2 + \lambda_m^2} \left(\frac{\kappa}{h} + \frac{I_0(\kappa a)}{I_1(\kappa a)} \right) \cdot$$

$$\cdot \left(1 - \alpha \kappa^2 \int_0^{t_1} n_1(t) e^{\alpha \lambda_m^2 t} dt\right) \Big] \Big\},$$

$$0 \leq r \leq a, \qquad 0 < h < \infty \quad (42)$$

Equation 42 is applicable to problems in which a steady state temperature distribution exists at $t < 0$ and in which it may be assumed that, at $t = 0$, the boundary conductance, which is expressed by $U = kh$, and the coolant temperature change instantaneously to new constant values.

If the boundary conductance and the coolant temperature are uniformly constant from $t = 0$ to $t = t_1$, an alternate form of equation 42 may be obtained by specifying the coolant temperature to be zero. By substituting equation 36 into equation 42 and collecting terms, a form convenient for computation is obtained

$$T_1(r,t_1) = \frac{\kappa Q_o}{\pi k a^2}\left(\frac{\kappa}{h} + \frac{I_o(\kappa a)}{I_1(\kappa a)}\right) \sum_{m=1}^{\infty} \frac{J_o(\lambda_m r)}{\lambda_m J_1(\lambda_m a)} \cdot$$

$$\cdot \frac{1}{1 + \frac{\lambda_m^2}{h^2}} \cdot \frac{1}{\kappa^2 + \lambda_m^2} \left[e^{-\alpha\lambda_m^2 t_1} + \alpha\lambda_m^2 \int_0^{t_1} n_1(t) \cdot\right.$$

$$\left. \cdot\, e^{\alpha\lambda_m^2 (t_1 - t)}\, dt\right], \quad 0 < h \tag{43}$$

It is worthwhile to note that in equation 43, as t_1 becomes large, the first term inside the brackets approaches zero and becomes negligible.

When evaluation is to be made at the surface $r = a$, equation 43 can be simplified by using the relation

$$J_o(\lambda_m a)/[J_1(\lambda_m a) \cdot \lambda_m] = \frac{1}{h}, \quad m = 1,2,3,..$$

which yields

$$T_1(r,t_1) = A \sum_{m=1}^{\infty} \frac{M(\lambda_m,r)}{\left(1 + \frac{\lambda_m^2}{h^2}\right)(\kappa^2 + \lambda_m^2)} \left[e^{-\alpha\lambda_m^2 t_1} +\right.$$

$$\left. + \alpha\lambda_m^2 \int_0^{t_1} n_1(t) \cdot e^{-\alpha\lambda_m^2 (t_1 - t)}\, dt\right], \quad 0<h \tag{44}$$

where

$$A = \frac{\kappa Q_0}{\pi k a^2}\left(\frac{\kappa}{h} + \frac{I_0(\kappa a)}{I_1(\kappa a)}\right), \quad \kappa > 0 \tag{45a}$$

$$A = \frac{2Q_0}{\pi k a^3}, \quad \kappa = 0 \tag{45b}$$

$$M(\lambda_m, r) = \frac{J_0(\lambda_m r)}{\lambda_m J_1(\lambda_m a)}, \quad r < a \tag{46a}$$

$$M(\lambda_m, r) = \frac{1}{h}, \quad r = a \tag{46b}$$

and λ_m is defined by equation 8a.

In numerical evaluation, it is often convenient to employ a dimensionless ratio,

$$R_1(r, t_1) = \frac{T_1(r, t_1) - T_c}{T_1(r, 0) - T_c} \tag{47}$$

where T_c is the average coolant temperature.

By Newton's law of cooling, an evaluation of R at the surface directly gives the ratio of the heat transfer from the element at $t = t_1$ to the steady state rate at time zero.

The limitation of equation 44, as κ approaches zero, is identical with the case of a heat generation ratio independent of radius. By the elementary thermal neutron diffusion theory, after Glasstone and Edlund (12), κ is the inverse thermal diffusion length. Thus, κ is inversely proportional to the square root of the mean square distance that a statistically average thermal neutron would travel from the point it just becomes thermal to the point of capture.

The case of constant boundary conductance and variable coolant temperature will now be evaluated.

Purely mathematical solutions of sufficient generality to problems involving a variable coolant temperature, in addition to a radius and time dependent heat generation rate, are extremely difficult to obtain. In order to provide a method of attack on those problems, an adaptation of the solution to the basic equation 4, appropriate for use in a step-wise method of calculation, will be developed.

In the development it will be assumed as an approximation that the coolant temperature undergoes an instantaneous change at each time division t_i, but the coolant temperature is constant throughout each time increment $\Delta t_i = t_i - t_{i-1}$. The difference between the coolant temperature for an increment Δt_i and the steady state coolant temperature is ϵ_i. Hence,

$$T_i(r,t_p) - T_{i-1}(r,t_p) = \epsilon_i - \epsilon_{i-1} \qquad (48)$$

where subscript i indicates that T is referred to the coolant temperature existing during the ith time increment.

Substitution of equation 8b for $J_1^2(\lambda_m a)$ into the equation 19b yields the following form

$$T_{i-1}(r,t_{i-1}) = \frac{2}{a^2} \sum_{m=1}^{\infty} \frac{\lambda_m^2 J_o(\lambda_m r)\, e^{-\alpha\lambda_m^2 t_{i-1}}}{(h^2 + \lambda_m^2)\, J_o^2(\lambda_m a)} \left\{ \bar{T}_{i-1}(r,t_{i-2}) \cdot e^{-\alpha\lambda_m^2 t_{i-2}} + \frac{q_o}{c\rho}\bar{m}(r) \int_{t_{i-2}}^{t_{i-1}} n_{i-1}(t) e^{\alpha\lambda_m^2 t} dt \right\}$$

(49)

The subscript (i-1) of $T_{i-1}(r,t_{i-1})$ and $\overline{T}_{i-1}(r,t_{i-2})$ in equation 49 indicates that they refer to the coolant temperatures corresponding to the (i-1)'th time increment.

By equation 48

$$T_i(r,t_{i-1}) = (\epsilon_i - \epsilon_{i-1}) + T_{i-1}(r,t_{i-1}) \qquad (50)$$

Hence, the finite Hankel transform $T_i(r,t_{i-1})$ is

$$\overline{T}_i(r,t_{i-1}) = \overline{j}_o(\epsilon_i - \epsilon_{i-1}) + \overline{T}_{i-1}(r,t_{i-1}) \qquad (51)$$

or, by the following form,

$$\overline{j}_o(\epsilon_i - \epsilon_{i-1}) = (\epsilon_i - \epsilon_{i-1})\int_0^a rJ_o(\lambda_m r)dr$$

$$= \frac{a}{\lambda_m}(\epsilon_i - \epsilon_{i-1})J_1(\lambda_m a) \qquad (52)$$

$$\overline{T}_i(r,t_{i-1}) = \frac{a}{\lambda_m}(\epsilon_i - \epsilon_{i-1})J_1(\lambda_m a) + \overline{T}_{i-1}(r,t_{i-1}) \qquad (53)$$

A comparison of the form of equation 49 with the inversion formula, equation 9, shows that

$$\overline{T}_i(r,t_{i-1}) = e^{-\alpha\lambda_m^2 t_{i-1}}\Big\{\overline{T}_{i-1}(r,t_{i-2})\, e^{\alpha\lambda_m^2 t_{i-2}} +$$

$$+ \frac{q_o}{c\rho}\,\overline{m}(r)\int_{t_{i-2}}^{t_{i-1}} n_{i-1}(t)\, e^{\alpha\lambda_m^2 t}\, dt\Big\} \qquad (54)$$

Thus, substitution of equation 54 into equation 51 results in

$$\overline{T}_i(r,t_{i-1}) = \frac{a}{\lambda_m}J_1(\lambda_m a)(\epsilon_i - \epsilon_{i-1}) + e^{-\alpha\lambda_m^2 \Delta t_{i-1}}\cdot\overline{T}_{i-1}(r,t_{i-2})$$

$$+ e^{-\alpha\lambda_m^2 t_{i-1}}\cdot\frac{q_o}{c\rho}\,\overline{m}(r)\int_{t_{i-2}}^{t_{i-1}} n_{i-1}(t)e^{\alpha\lambda_m^2 t}\, dt \qquad (55)$$

Letting the index i in equation 55 take on the values 1, 2, 3, ... generates a sequence of transformation equations corresponding to the successive time increments Δt_1, Δt_2, Δt_3, ... Equation 56a is obtained directly from equation 53

$$\bar{T}_1(r,0) = \bar{T}_0(r,0) + \frac{a}{\lambda_m} J_1(\lambda_m a) \cdot \epsilon_1 \qquad (56a)$$

and from equation 55

$$\bar{T}_2(r,t_1) = \frac{a}{\lambda_m} J_1(\lambda_m a)(\epsilon_2 - \epsilon_1) + e^{-\alpha\lambda_m^2 \Delta t_1} \cdot \bar{T}_1(r,0)$$
$$+ e^{-\alpha\lambda_m^2 t_1} \cdot \frac{q_0}{c\rho} \bar{m}(r) \int_0^{t_1} n_1(t) e^{\alpha\lambda_m^2 t} \, dt \qquad (56b)$$

$$\bar{T}_3(r,t_2) = \frac{a}{\lambda_m} J_1(\lambda_m a)(\epsilon_3 - \epsilon_2) + e^{-\alpha\lambda_m^2 \Delta t_2} \cdot \bar{T}_2(r,t_1)$$
$$+ e^{-\alpha\lambda_m^2 t_2} \cdot \frac{q_0}{c\rho} \bar{m}(r) \int_{t_1}^{t_2} n_2(t)\, e^{\alpha\lambda_m^2 t} \, dt \qquad (56c)$$

$$\bar{T}_4(r,t_3) = \frac{a}{\lambda_m} J_1(\lambda_m a)(\epsilon_4 - \epsilon_3) + e^{-\alpha\lambda_m^2 \Delta t_3} \cdot \bar{T}_3(r,t_2)$$
$$+ e^{-\alpha\lambda_m^2 t_3} \cdot \frac{q_0}{c\rho} \bar{m}(r) \int_{t_2}^{t_3} n_3(t)\, e^{\alpha\lambda_m^2 t} \, dt \qquad (56d)$$

etc.

By specifing $\epsilon_0 = t_0 = 0$ and substituting each equation in the sequence into the one immediately following, it results, after collection of terms, in the following transformation equation relating $\bar{T}_i(r,t_{i-1})$ to $\bar{T}_0(r,0)$,

$$\bar{T}_i(r,t_{i-1}) = e^{-\alpha\lambda_m^2 t_{i-1}} \left\{ \frac{a}{\lambda_m} J_1(\lambda_m a) \sum_{p=0}^{p=i-1} (\epsilon_{p+1} - \epsilon_p) \cdot e^{\alpha\lambda_m^2 t_p} \right.$$

$$\left. + \bar{T}_o(r,0) + \frac{q_o}{c\rho} \bar{m}(r) \int_{t=0}^{t=i-1} n_{i-1}(t) e^{\alpha\lambda_m^2 t} dt \right\} \quad (57)$$

However, by assuming that the initial coolant temperature for $t < 0$ is zero and that the boundary conductance expressed by $U = kh$ is uniformly constant, substitution of equations 36, 39c, 40 and 57 into equation 19b results, after rearrangement and collection of terms, in

$$T_i(r,t_i) = A \sum_{m=1}^{\infty} \frac{M(\lambda_m,r)}{(1 + \frac{\lambda_m^2}{h^2})(\kappa^2 + \lambda_m^2)} \left[e^{-\alpha\lambda_m^2 t_i} + \frac{2}{aA}(\kappa^2+\lambda_m^2) \right.$$

$$\left. \cdot \sum_{p=0}^{p=i-1} (\epsilon_{p+1} - \epsilon_p) e^{-\alpha\lambda_m^2 (t_i - t_p)} + \alpha\lambda_m^2 \int_{t=0}^{t=i} n_i(t) e^{-\alpha\lambda_m^2 (t_i - t)} dt \right],$$

$$0 < h \quad (58)$$

The quantities A, $M(\lambda_m,r)$ and λ_m have been previously defined by equations 45a and 45b, 46a and 46b, and 8a respectively. The subscript i appended to $T(r,t_i)$ indicates that T is measured above the coolant temperature existing during the time increment $\Delta t_i = t_i - t_{i-1}$.

Equation 58 is applicable, for step-wise computation, to problems for which the boundary conductance remains constant and the coolant temperature varies arbitrary from a zero steady state value at time $t = 0$.

For the case of uniform heat generation, i.e., = 0, equation 58 simplifies as does equation 44.

The heat transfer rate from the surface of a fuel element is radially evaluated with equation 58 by employing Newton's law of cooling. Therefore,

$$Q(t) = 2\pi aU\,[T(a,t) - T_c]$$

$$= 2\pi aUT(a,t), \quad T_c = 0 \qquad (59)$$

COMPUTER PROGRAMMING AND RESULTS

In order to apply the derived equations in predicting temperature distributions in a superheating region of the long solid cylindrical fuel element in nuclear power reactors during the transient state after a sudden power reduction, equation 58 has been programmed on an IBM-360 digital computer.

The form of equation 58 in the program is

$$T_1(r,t_1) = A \sum_{m=1}^{\infty} \frac{M(\lambda_m,r)e^{-\alpha\lambda_m^2 t_1}}{(1+\frac{\lambda_m^2}{h^2})(\kappa^2+\lambda_m^2)} [1 + \frac{2}{aA}(\kappa^2 + \lambda_m^2) \cdot$$

$$\cdot \sum_{p=0}^{p=i-1} (\epsilon_{p+1} - \epsilon_p)e^{-\alpha\lambda_m^2 t_p} + \alpha\lambda_m^2 \int_{t=0}^{t=1} n_1(t)e^{\alpha\lambda_m^2 t} dt], 0<h.$$

where

$$A = \frac{\kappa Q_o}{\pi k a^2}(\frac{\kappa}{h} + \frac{I_o(\kappa a)}{I_1(\kappa a)})$$

$$M(\lambda_m,r) = \frac{J_o(\lambda_m r)}{\lambda_m J_1(\lambda_m a)} .$$

Given Data

Natural uranium was chosen as the fuel material and saturated steam was chosen as the coolant. Material properties were evaluated at 100° C, 200° C, 300° C, 400° C and 600° C, which were assumed as mean temperatures of the fuel material within time intervals, 30sec-50sec, 17sec-30sec, 9sec-17sec, 3sec-9sec, and 0sec-3sec, respectively. Physical properties of the fuel element and the coolant were assumed as follows: the radius of the fuel element, a = 0.500 inch in one case and a = 0.336 inch in another case; h = 3.268013 in^{-1}; mean energy released per fission, G, and statistical mean neutron flux, ϕ, are 200 Mev/fission and 1×10^{14} neutrons/cm^2 sec, respectively.

The quantity, n(t)*, is given by a function by which the heat generation rate varies with time,

$$n(t) = 1 - (0.251t^3 - 0.219t^4) \qquad , 0 \leq t \leq 1$$

$$n(t) = 0.2369/(t - 0.9305)^{0.302} \qquad , 1 \leq t \leq 3$$

$$n(t) = 0.2805/(t - 0.3250)^{0.3685} \qquad , 3 \leq t \leq 1000$$

where t is expressed in seconds.

Values of $\epsilon_{p+1} - \epsilon_p$, expressed in equation 50 at different assumed-average temperatures of the fuel element, are taken as in Table 1.

Other properties (16), taking mean values, are also tabulated in Table 1.

Calculated Data and Results

The total heat transmission rate per unit length, Q_o, and macroscopic fission cross section, Σ_f, were calculated at different temperatures, after Benedict and Pigford (14)

*The selected equations were obtained by M. B. Larson, Oregon State University, Corvallis, Oregon.

and El-Wakil (15), and tabulated in Table 1 in both cases of $a = 0.500$ inch and $a = 0.336$ inch.

The distance from the central line of the fuel element, r, was chosen as $r/a = 0$, $r/a = 1/4$, $r/a = 1/2$, $r/a = 3/4$, and $r/a = 1$ in case of $a = 0.500$ inch and $r/a = 0$, $r/a = 1/5$, $r/a = 2/5$, $r/a = 3/5$, $r/a = 4/5$, and $r/a = 1$ in case of $a = 0.336$ inch.

Equation 8b was programmed on the computer. The values of λ_m were calculated from the computer results and tabulated in Table 2. The function n(t) on which the heat generation rate depends as a function of time was also calculated by the computer. The values of n(t) at each time unit has been tabulated in Table 3.

The integral $\int_{t=0}^{t=i} n(t)\, e^{\alpha \lambda_m^2 t}\, dt$ was programmed on the computer by using the trapezoidal rule, after Hildebrand (11).

By using the given and calculated data, temperature distributions in a superheating region of the long solid cylindrical fuel element and the heat transmission rate from the surface of the fuel material were determined and are tabulated in Table 4 through Table 7. They are also plotted in Figure 3 through Figure 7 with various temperature structures at both cases of $a = 0.500$ inch and $a = 0.336$ inch and then the results are compared with each other.

Table 1. The properties of fuel material used

T_m(°C)	α(in^2/sec)	(in^{-1})	k(Btu/hr·ft·°F)	Σ_f(cm^{-1})	$\epsilon_{p+1}-\epsilon_p$(°F) 1	Q_o(Btu/hr·ft) a=0.50 in	Q_o(Btu/hr·ft) a=0.336 in
100	0.01757	1.958	16.20	0.150	14	$2.276 \cdot 10^5$	$1.023 \cdot 10^5$
200	0.01755	1.781	17.42	0.136	15	$2.065 \cdot 10^5$	$0.933 \cdot 10^5$
300	0.01753	1.674	18.14	0.124	16	$1.872 \cdot 10^5$	$0.845 \cdot 10^5$
400	0.01751	1.654	18.82	0.114	18	$1.747 \cdot 10^5$	$0.787 \cdot 10^5$
600	0.01594	1.552	20.00	0.100	27	$1.519 \cdot 10^5$	$0.684 \cdot 10^5$

Table 2. The values of λ_m with m = 1, 2, 3, 4, 5

a (inch)	λ_m (ft^{-1})				
	λ_1	λ_2	λ_3	λ_4	λ_5
0.500	36.00	101.25	173.60	248.00	322.50
0.336	46.60	146.60	256.80*		

*First trials show that λ_m behind m = 3 can be neglected.

Table 3. The function of n(t)

Time (sec)	Function n(t)
0.00	1.00000
0.20	0.99764
0.40	0.97833
0.60	0.91740
0.80	0.78179
1.00	0.53001
1.50	0.28081
2.00	0.23214
2.50	0.20675
3.00	0.19519
5.00	0.15890
7.00	0.13936
9.00	0.12653
11.00	0.11721
13.00	0.11003
15.00	0.10424
17.00	0.09944
19.00	0.09538
21.00	0.09187
23.00	0.08880
25.00	0.08607
30.00	0.08041
35.00	0.07593
40.00	0.07225
45.00	0.06916
50.00	0.06651

Table 4. Transient temperature distributions as a function of time as well as radial position (material properties evaluated at 400° C)

t (sec)	r/a = 0		r/a = 1/4		r/a = 1/2		r/a = 3/4		r/a = 1	
	T (°F)	T/T_o	T (°F)	T/T_o	T (°F)	T/T_o	T (°F)	T/T_o	T (°F)	T/T_o
0.0	1786	1.000	1737	1.000	1596	1.000	1355	1.000	1006	1.000
0.2	1768	0.990	1719	0.990	1578	0.989	1337	0.987	990	0.985
0.4	1768	0.990	1719	0.990	1578	0.989	1337	0.987	990	0.985
0.6	1767	0.989	1719	0.989	1578	0.989	1336	0.986	990	0.984
0.8	1762	0.987	1713	0.987	1572	0.985	1331	0.982	984	0.978
1.0	1748	0.979	1700	0.979	1558	0.976	1316	0.971	971	0.965
1.5	1686	0.944	1637	0.943	1494	0.936	1252	0.924	919	0.914
2.0	1607	0.900	1558	0.897	1415	0.887	1180	0.871	833	0.858
2.5	1524	0.854	1476	0.850	1337	0.838	1111	0.820	811	0.807
3.0	1442	0.808	1396	0.804	1262	0.791	1046	0.772	763	0.759
5.0	1144	0.641	1104	0.636	996	0.624	825	0.609	608	0.604
7.0	908	0.509	876	0.505	791	0.496	656	0.485	484	0.481
9.0	728	0.408	702	0.405	635	0.398	528	0.390	390	0.388
11.0	591	0.331	570	0.328	514	0.324	430	0.317	319	0.317
13.0	488	0.273	470	0.271	426	0.267	356	0.263	265	0.263
15.0	409	0.229	394	0.227	358	0.224	299	0.221	223	0.222
17.0	348	0.195	336	0.193	306	0.191	256	0.189	192	0.191
19.0	302	0.169	291	0.168	266	0.166	223	0.165	168	0.167
21.0	267	0.149	257	0.148	235	0.147	198	0.146	149	0.148
23.0	239	0.134	231	0.133	211	0.132	178	0.131	135	0.139
25.0	218	0.122	210	0.121	192	0.120	162	0.120	123	0.123
30.0	177	0.099	169	0.097	159	0.099	140	0.103	114	0.113
35.0	157	0.088	150	0.086	141	0.088	125	0.092	102	0.101
40.0	144	0.081	137	0.079	129	0.081	115	0.085	94	0.094
45.0	135	0.075	128	0.074	121	0.076	108	0.079	89	0.088
50.0	128	0.072	122	0.070	115	0.072	102	0.076	84	0.084

Table 5. Transient temperature distribution as a function of time as well as radial position (material properties evaluated at 600° C)

t i(sec)	r/a = 0 T (°F)	r/a = 0 T/T$_o$	r/a = 1/4 T (°F)	r/a = 1/4 T/T$_o$	r/a = 1/2 T (°F)	r/a = 1/2 T/T$_o$	r/a = 3/4 T (°F)	r/a = 3/4 T/T$_o$	r/a = 1 T (°F)	r/a = 1 T/T$_o$
0.0	1510	1.000	1467	1.000	1349	1.000	1146	1.000	853	1.000
0.2	1483	0.982	1442	0.983	1323	0.981	1120	0.977	829	0.972
0.4	1483	0.982	1442	0.983	1323	0.981	1120	0.977	829	0.972
0.6	1483	0.982	1442	0.982	1322	0.980	1120	0.977	828	0.971
0.8	1479	0.979	1438	0.980	1318	0.977	1115	0.973	825	0.967
1.0	1468	0.973	1428	0.973	1308	0.970	1105	0.964	815	0.955
1.5	1421	0.941	1380	0.940	1259	0.934	1056	0.921	775	0.909
2.0	1361	0.901	1320	0.899	1199	0.889	1000	0.873	732	0.858
2.5	1298	0.859	1257	0.857	1139	0.844	946	0.826	691	0.811
3.0	1234	0.818	1195	0.814	1081	0.801	896	0.782	654	0.767
5.0	1000	0.662	965	0.658	870	0.645	721	0.629	529	0.621
7.0	808	0.535	780	0.532	704	0.522	583	0.509	429	0.503
9.0	658	0.436	635	0.433	574	0.425	476	0.415	350	0.411
11.0	541	0.358	522	0.356	472	0.350	392	0.342	289	0.339
13.0	450	0.298	434	0.256	393	0.291	327	0.285	242	0.284
15.0	379	0.251	366	0.249	331	0.246	276	0.241	205	0.241
17.0	324	0.214	312	0.213	284	0.210	237	0.207	176	0.207
19.0	281	0.186	271	0.184	246	0.182	206	0.180	154	0.180
21.0	247	0.163	238	0.162	217	0.161	182	0.159	136	0.160
23.0	220	0.146	212	0.145	194	0.144	163	0.142	122	0.143
25.0	199	0.132	192	0.131	175	0.130	148	0.129	111	0.130
30.0	159	0.105	152	0.104	142	0.105	123	0.108	99	0.116
35.0	138	0.091	132	0.090	123	0.092	108	0.094	87	0.102
40.0	125	0.083	119	0.081	112	0.083	98	0.086	79	0.093
45.0	116	0.077	111	0.075	104	0.077	91	0.080	74	0.087
50.0	109	0.072	104	0.071	98	0.073	86	0.075	70	0.082

Table 6. Transient temperature distributions as a function of time as well as radial position, a = 0.500 inch (material properties are evaluated at: 600°C, 0≤t≤3; 400°C, 3≤t≤9; 300°C, 9≤t≤17; 200°C, 17≤t≤30; 100°C, 30≤t≤50)

t (sec)	r/a = 0		r/a = 1/4		r/a = 1/2		r/a = 3/4		r/a = 1	
	T (°F)	T/T_o	T (°F)	T/T_o	T (°F)	T/T_o	T (°F)	T/T_o	T (°F)	T/T_o
0.000	1353	1.0000	1313	1.0000	1205	1.0000	1028	1.0000	761	1.0000
0.001	1325	0.9792	1287	0.9801	1180	0.9791	1001	0.9736	738	0.9698
0.005	1326	0.9800	1288	0.9808	1181	0.9799	1002	0.9746	739	0.9710
0.010	1327	0.9808	1289	0.9816	1182	0.9807	1003	0.9756	740	0.9721
0.020	1328	0.9816	1290	0.9824	1183	0.9815	1004	0.9767	741	0.9732
0.040	1330	0.9824	1291	0.9831	1184	0.9823	1005	0.9776	742	0.9744
0.060	1331	0.9832	1292	0.9839	1185	0.9831	1006	0.9786	743	0.9755
0.080	1332	0.9839	1293	0.9847	1186	0.9840	1007	0.9796	744	0.9766
0.100	1333	0.9847	1294	0.9854	1187	0.9848	1008	0.9806	744	0.9776
0.150	1334	0.9854	1295	0.9862	1188	0.9856	1009	0.9815	745	0.9786
0.200	1335	0.9861	1296	0.9869	1189	0.9864	1010	0.9824	746	0.9795
0.250	1336	0.9868	1297	0.9877	1190	0.9873	1011	0.9833	746	0.9804
0.300	1336	0.9875	1298	0.9884	1191	0.9881	1012	0.9841	747	0.9812
0.400	1337	0.9881	1299	0.9890	1191	0.9887	1012	0.9846	747	0.9816
0.600	1337	0.9876	1298	0.9886	1191	0.9881	1011	0.9835	746	0.9801
0.800	1333	0.9849	1295	0.9860	1187	0.9852	1007	0.9794	743	0.9752
1.000	1324	0.9779	1286	0.9791	1178	0.9775	997	0.9694	734	0.9638
1.500	1280	0.9460	1243	0.9465	1134	0.9411	952	0.9262	698	0.9167

Table 6. (Continued)

t 1(sec)	r/a = 0 T (°F)	r/a = 0 T/T_o	r/a = 1/4 T (°F)	r/a = 1/4 T/T_o	r/a = 1/2 T (°F)	r/a = 1/2 T/T_o	r/a = 3/4 T (°F)	r/a = 3/4 T/T_o	r/a = 1 T (°F)	r/a = 1 T/T_o
2.000	1226	0.9060	1189	0.9052	1080	0.8963	902	0.8770	659	0.8653
2.500	1169	0.8640	1133	0.8623	1026	0.8513	853	0.8300	622	0.8174
3.000	1112	0.8220	1077	0.8198	973	0.8077	808	0.7857	589	0.7730
5.000	1053	0.7779	1015	0.7730	914	0.7584	761	0.7398	556	0.7305
7.000	836	0.6175	806	0.6135	726	0.6022	605	0.5883	443	0.5815
9.000	670	0.4950	646	0.4917	582	0.4829	486	0.4729	356	0.4682
11.000	603	0.4456	581	0.4426	524	0.4350	439	0.4272	323	0.4238
13.000	497	0.3672	479	0.3646	432	0.3588	363	0.3534	267	0.3513
15.000	416	0.3076	401	0.3054	362	0.3008	306	0.2973	226	0.2962
17.000	406	0.2997	391	0.2975	353	0.2934	299	0.2912	222	0.2910
19.000	352	0.2600	339	0.2580	307	0.2547	261	0.2537	193	0.2541
21.000	310	0.2294	299	0.2276	271	0.2249	231	0.2248	172	0.2256
23.000	278	0.2058	268	0.2041	243	0.2019	208	0.2024	155	0.2035
25.000	253	0.1873	244	0.1857	222	0.1839	190	0.1849	142	0.1862
30.000	207	0.1532	197	0.1502	181	0.1503	166	0.1612	129	0.1695
35.000	216	0.1597	205	0.1563	189	0.1568	175	0.1704	137	0.1804
40.000	198	0.1465	188	0.1433	173	0.1439	161	0.1570	127	0.1667
45.000	186	0.1372	176	0.1341	162	0.1348	152	0.1475	119	0.1569
50.000	176	0.1302	167	0.1272	154	0.1280	144	0.1402	114	0.1493

Table 7. Transient temperature distributions as a function of time as well as radial position, a = 0.336 inch (material properties are evaluated at: 600°C, $0 \leq t \leq 3$; 400°C, $3 \leq t \leq 9$; 300°C, $9 \leq t \leq 17$; 200°C, $17 \leq t \leq 30$; 100°C, $30 \leq t \leq 50$)

t (sec)	r/a = 0		r/a = 1/5		r/a = 2/5		r/a = 3/5		r/a = 4/5		r/a = 1	
	T (°F)	T/T_o	T (°F)	T/T_o	T (°F)	T/T_o	T (°F)	T/T_o	T (°F)	T/T_o	T (°F)	T/T_o
0.000	784	1.000	772	1.000	739	1.000	687	1.000	614	1.000	513	1.000
0.001	757	0.965	746	0.966	714	0.966	661	0.962	587	0.967	490	0.954
0.005	758	0.967	747	0.967	715	0.967	662	0.964	588	0.958	491	0.956
0.010	759	0.968	748	0.969	716	0.969	663	0.965	589	0.960	491	0.957
0.020	760	0.969	749	0.970	717	0.970	664	0.967	590	0.962	492	0.959
0.040	761	0.971	750	0.971	718	0.971	665	0.968	591	0.963	493	0.961
0.060	762	0.972	751	0.972	719	0.973	666	0.970	592	0.965	494	0.962
0.080	763	0.973	752	0.974	720	0.974	667	0.971	593	0.966	495	0.964
0.100	764	0.974	753	0.975	721	0.975	668	0.973	594	0.968	496	0.966
0.150	765	0.976	754	0.976	722	0.977	669	0.974	595	0.969	496	0.967
0.200	766	0.977	755	0.978	723	0.978	670	0.975	596	0.971	497	0.968
0.250	767	0.978	756	0.979	724	0.979	671	0.977	596	0.972	498	0.969
0.300	768	0.979	757	0.980	725	0.980	672	0.978	597	0.973	498	0.971
0.400	769	0.980	758	0.981	725	0.981	673	0.979	598	0.974	499	0.971
0.600	768	0.980	757	0.980	724	0.980	671	0.977	596	0.972	497	0.969
0.800	764	0.975	753	0.975	720	0.975	667	0.971	592	0.965	494	0.962
1.000	755	0.962	743	0.963	711	0.962	657	0.957	582	0.949	485	0.945
1.500	710	0.906	699	0.905	667	0.902	614	0.893	542	0.883	450	0.877
2.000	656	0.837	645	0.836	614	0.831	565	0.822	497	0.810	413	0.805

Table 7. (Continued)

t (sec)	r/a = 0		r/a = 1/5		r/a = 2/5		r/a = 3/5		r/a = 4/5		r/a = 1	
	T (°F)	T/T_o	T (°F)	T/T_o	T (°F)	T/T_o	T (°F)	T/T_o	T (°F)	T/T_o	T (°F)	T/T_o
2.500	603	0.769	593	0.768	564	0.764	518	0.754	456	0.743	379	0.738
3.000	554	0.706	545	0.705	518	0.701	476	0.692	418	0.682	348	0.677
5.000	451	0.575	442	0.573	420	0.569	388	0.565	345	0.563	289	0.563
7.000	323	0.412	317	0.411	301	0.408	279	0.406	249	0.406	209	0.407
9.000	242.	0.309	238	0.308	226	0.306	210	0.305	188	0.307	158	0.308
11.000	211	0.269	207	0.268	197	0.266	183	0.267	165	0.270	139	0.271
13.000	174	0.222	170	0.221	162	0.219	151	0.220	137	0.223	116	0.225
15.000	149	0.190	146	0.189	139	0.188	130	0.189	118	0.192	100	0.194
17.000	151	0.193	148	0.192	141	0.191	132	0.192	121	0.196	102	0.199
19.000	138	0.176	135	0.175	128	0.174	121	0.175	110	0.180	93	0.182
21.000	128	0.163	125	0.162	119	0.161	112	0.163	103	0.167	87	0.169
23.000	121	0.154	118	0.153	112	0.152	106	0.154	97	0.158	82	0.160
25.000	115	0.146	112	0.145	107	0.145	101	0.147	92	0.151	78	0.153
30.000	105	0.134	102	0.133	98	0.132	96	0.140	94	0.153	82	0.160
35.000	117	0.150	114	0.148	109	0.147	107	0.156	105	0.171	92	0.179
40.000	112	0.142	109	0.141	104	0.140	102	0.149	100	0.163	88	0.170
45.000	107	0.136	104	0.134	99	0.134	97	0.142	95	0.156	84	0.163
50.000	102	0.130	99	0.129	95	0.128	94	0.136	92	0.149	80	0.156

Table 8. Normalized surface heat transmission rate per unit length (material properties are evaluated at: 600°C, $0 \leq t \leq 3$; 400°C, $3 \leq t \leq 9$; 300°C, $9 \leq t \leq 17$; 200°C, $17 \leq t \leq 30$; 100°C, $30 \leq t \leq 50$)

t	Q(t)/Q(0)			
	$\kappa > 0$		$\kappa = 0$	
(sec)	a=0.5 in	a=0.336 in	a=0.5 in	a=0.336 in
0.000	1.0000	1.0000	1.00000	1.00000
0.001	0.9698	0.954	0.96979	0.95375
0.005	0.9710	0.956	0.97095	0.95552
0.010	0.9721	0.957	0.97241	0.95729
0.020	0.9732	0.959	0.97325	0.95902
0.040	0.9744	0.961	0.97437	0.96069
0.060	0.9755	0.962	0.97547	0.96234
0.080	0.9766	0.964	0.97656	0.96396
0.100	0.9776	0.966	0.97764	0.96557
0.150	0.9786	0.967	0.97858	0.96690
0.200	0.9795	0.968	0.97950	0.96820
0.250	0.9804	0.968	0.98039	0.96947
0.300	0.9812	0.971	0.98124	0.97168
0.400	0.9816	0.971	0.98165	0.97117
0.600	0.9801	0.968	0.98016	0.96896
0.800	0.9852	0.962	0.97547	0.96179
1.000	0.9638	0.945	0.96453	0.94537
1.500	0.9167	0.877	0.91878	0.89840
2.000	0.8653	0.805	0.86836	0.80581
2.500	0.8174	0.738	0.82087	0.73885
3.000	0.7730	0.677	0.77664	0.67808
5.000	0.7305	0.563	0.73378	0.56234
7.000	0.5815	0.407	0.58395	0.40660
9.000	0.4682	0.308	0.46987	0.30730
11.000	0.4238	0.271	0.42509	0.27044
13.000	0.3513	0.225	0.35213	0.22429
15.000	0.2962	0.194	0.29666	0.19361
17.000	0.2910	0.189	0.29121	0.18449
19.000	0.2541	0.182	0.25405	0.18288
21.000	0.2256	0.169	0.22541	0.16853
23.000	0.2035	0.160	0.20318	0.15918
25.000	0.1862	0.153	0.18579	0.15184
30.000	0.1695	0.160	0.16643	0.15717
35.000	0.1804	0.179	0.17623	0.17564
40.000	0.1667	0.170	0.16268	0.16750
45.000	0.1569	0.163	0.15295	0.15953
50.000	0.1493	0.156	0.14547	0.15311

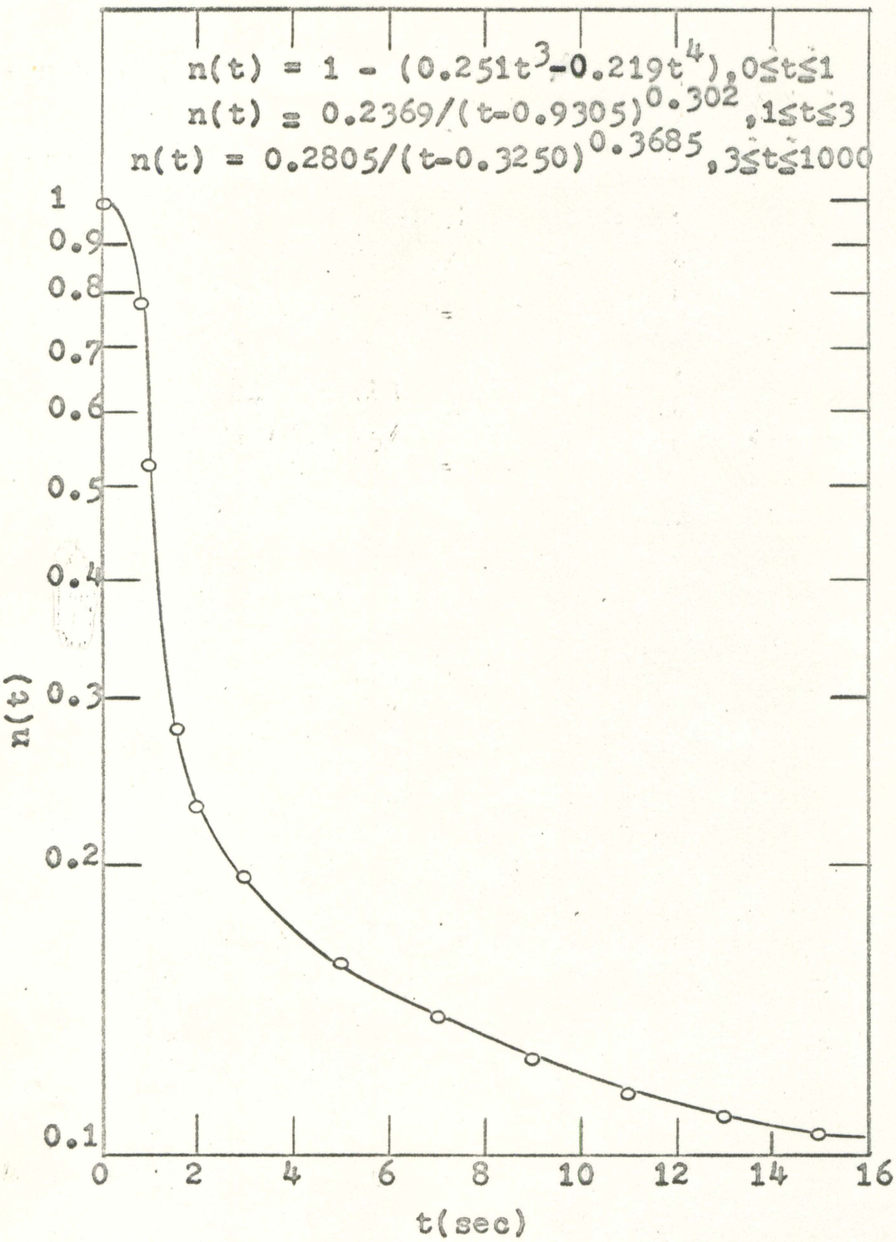

Figure 2. The function of n(t)

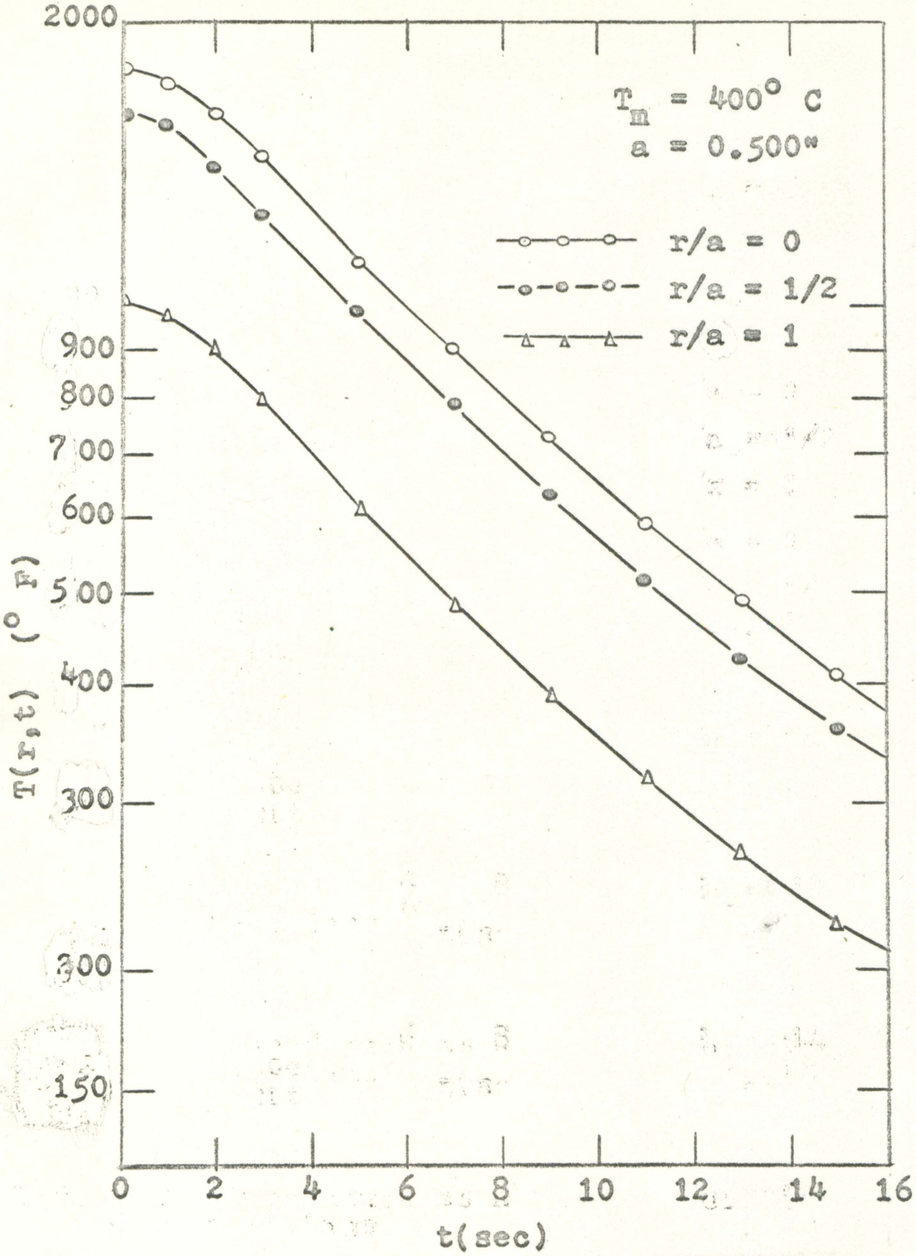

Figure 3. Temperature as a function of time at different positions

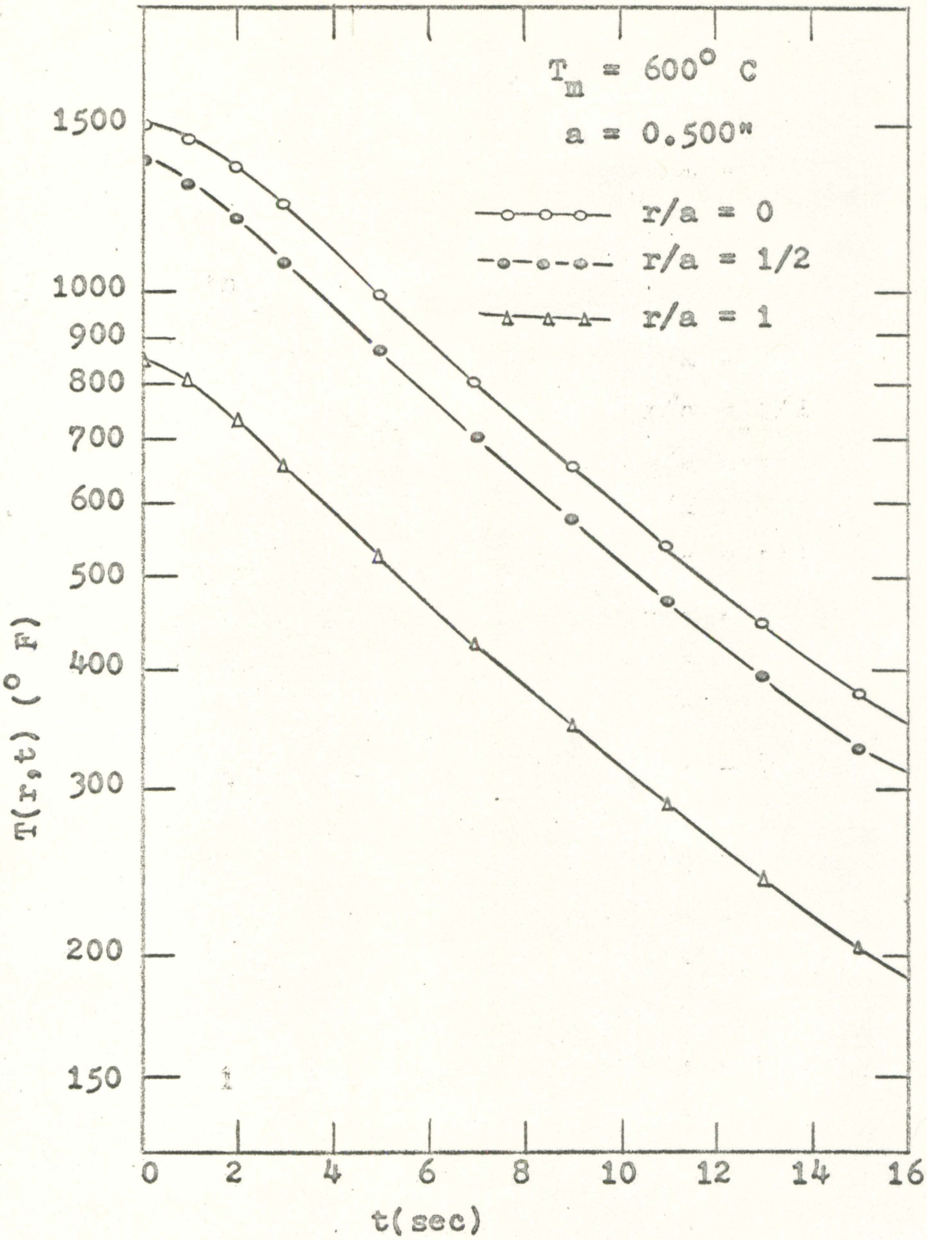

Figure 4. Temperature as a function of time at different positions

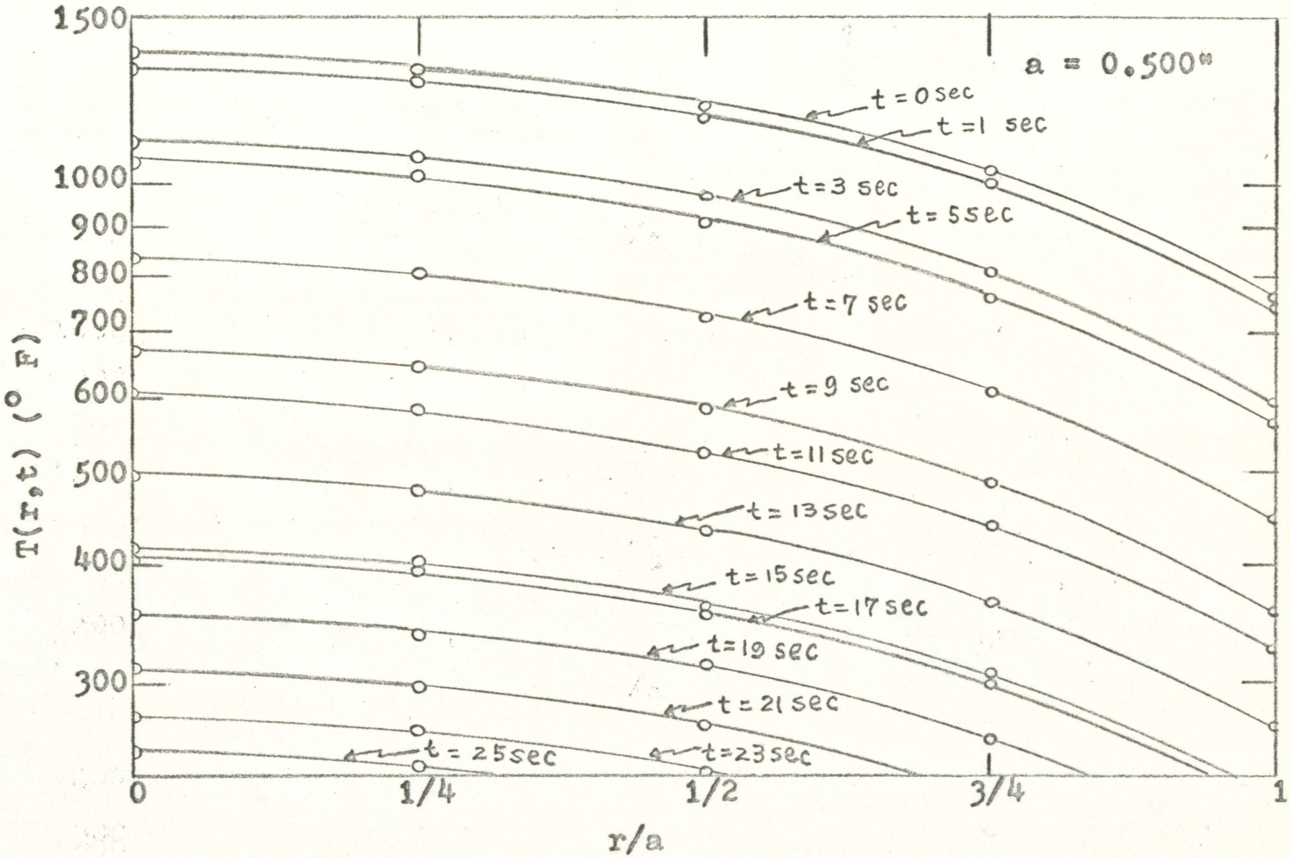

Figure 5. Temperature as a function of radius positions at various time

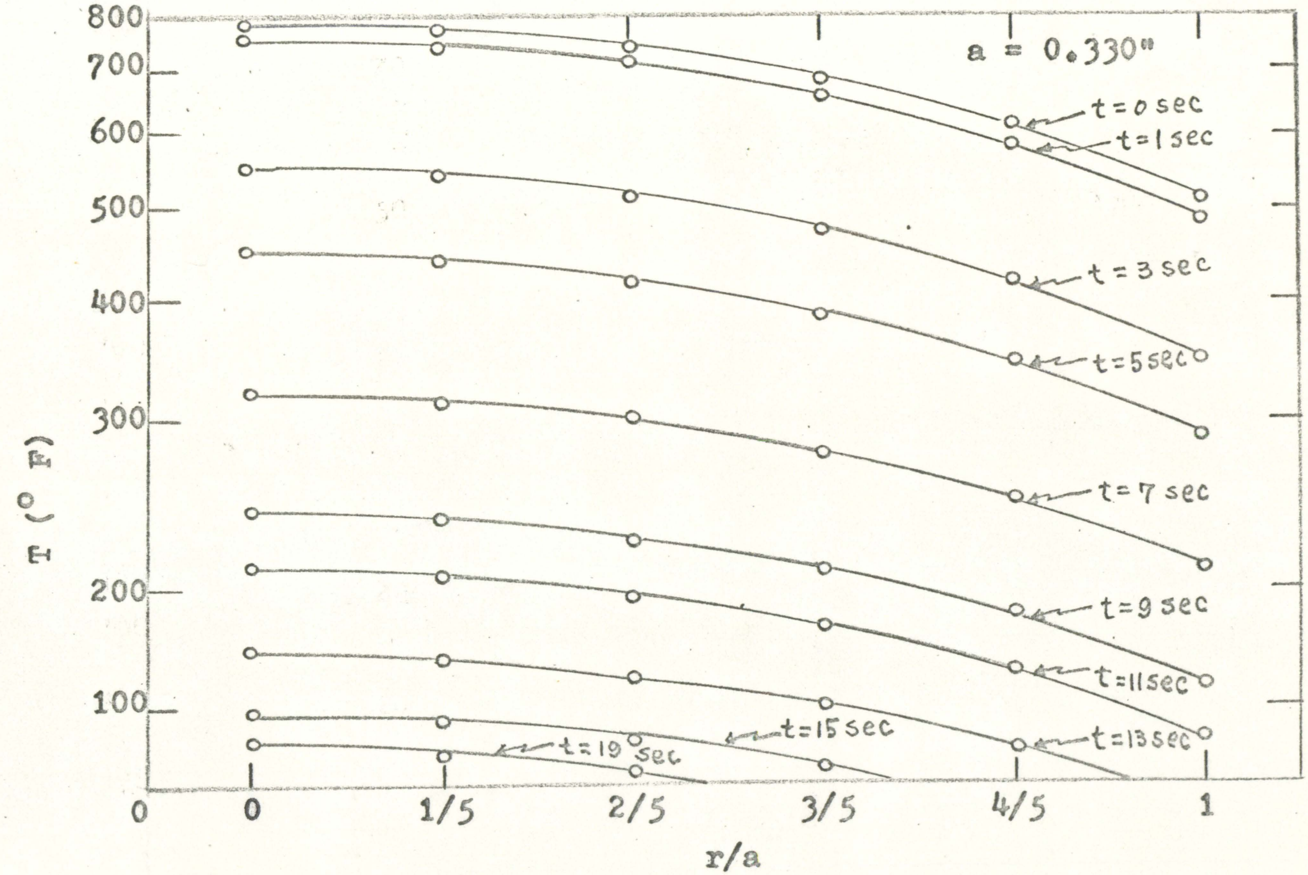

Figure 6. Temperature as a function of radius positions at various time

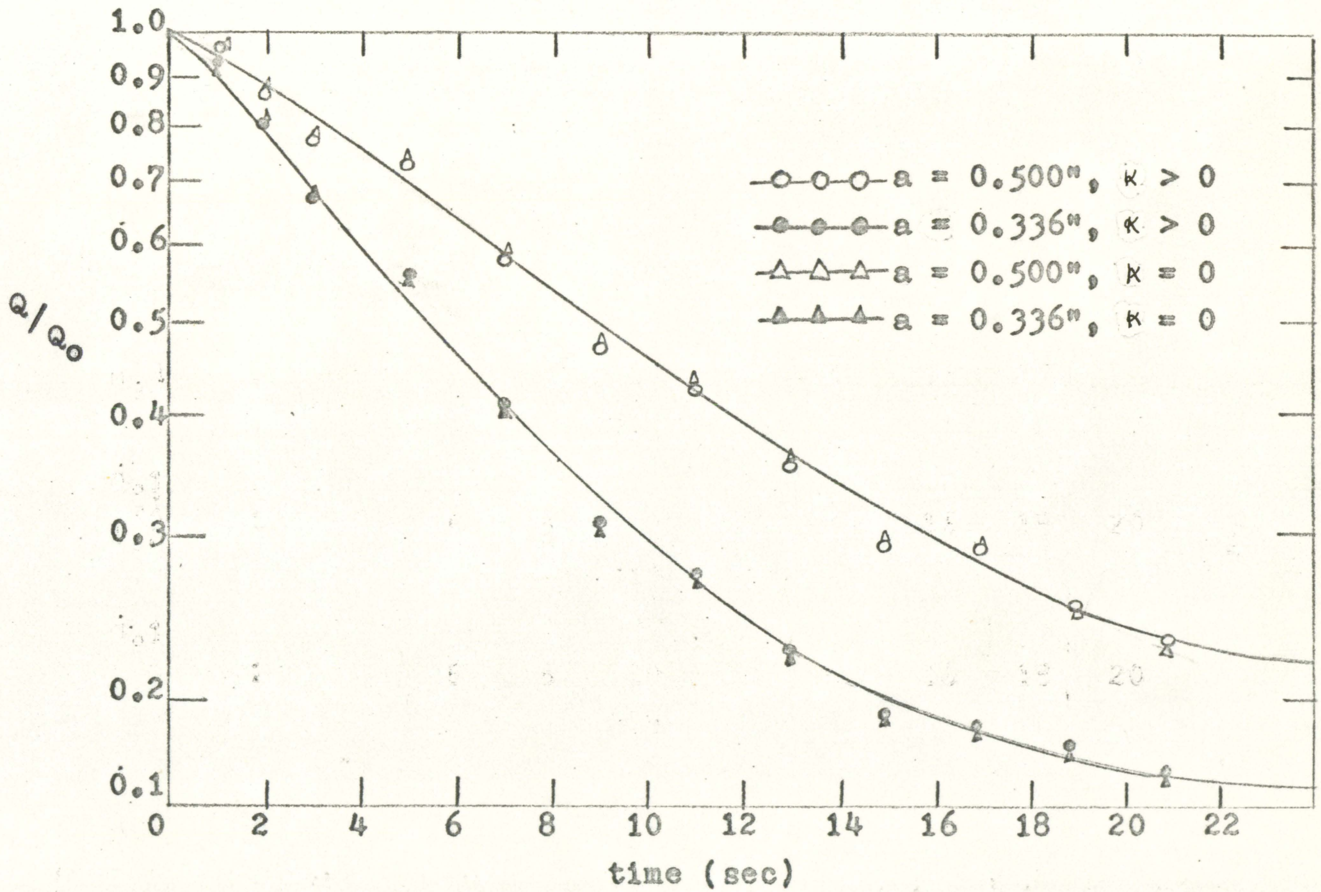

Figure 7. Heat transmission rate as a function of time

DISCUSSION OF RESULTS

Temperatures as a function of time as well as radial position are tabulated in Table 4 and plotted in Figure 3, in which material properties are evaluated at 400° C throughout the fuel element. In Table 5 and Figure 4, material properties are evaluated at 600° C.

From Figures 3 and 4, it is seen that temperatures decrease smoothly. The temperature distribution curve at $r/a = 1/2$ lies close to that at $r/a = 0$ as was expected.

It is seen from both Tables 4 and 5 at $r/a = 0$, $t = 0$ second, that the temperatures are more than 600° C and those at $r/a = 1$, $t = 50$ seconds, are less than 100° C. Therefore, material properties are next evaluated at 600° C, $0 \leq t \leq 3$; 400° C, $3 \leq t \leq 9$; 300° C, $9 \leq t \leq 17$; 200° C, $17 \leq t \leq 30$; 100° C, $30 \leq t \leq 50$, according to the results in Tables 4 and 5. Two cases are considered, i.e., $a = 0.500$ inch and $a = 0.336$ inch. The computer results are tabulated in Tables 6 and 7 and plotted in Figures 5 and 6, respectively. In both figures, $r/a = 0$ refers to the center of the fuel element and $r/a = 1$ refers to the surface of the fuel element.

Temperatures in Figure 5, which are shown for selected times, are functions of radial position and time. The figure shows that the differences of temperatures obtained at $r/a = 0$ and $r/a = 1$ are approximately 600° F from $t = 0$

second to t = 3 seconds, 200° F from t = 3 seconds to t = 17 seconds and less than 100° F for t > 17 seconds. The assumed temperatures for the material in the stipulated time intervals is responsible for the relatively wrong curve.

In Figure 6 intervals of radial distance are given as one fifth of the radius of the fuel element. Values of Q_o along the distance are re-evaluated and λ_m is calculated again according to a = 0.336 inch. With these newly evaluated properties of material, equation 58 is programmed on the computer again. The results of temperature distributions in Figure 6 show that the differences of temperatures from r/a = 0 and r/a = 1 are about 250° F from t = 0 second to t = 1.5 seconds, 100° F from t = 1.5 seconds to t = 9 seconds, 50° F from t = 9 seconds to t = 19 seconds, and then temperatures at r/a = 0 and r/a = 1 become very close with differences of about 20° F.

From Figures 5 and 6 it is evident that temperatures are very dependent upon the radius of the fuel element as well as the time, and it is seen that temperatures of the fuel element with a = 0.336 inch are much lower at any given time than those for a = 0.500 inch.

In Table 8, values of Q(t) are calculated by using equation 59, assuming T_c = 0, i.e., all temperatures are related to the coolant temperature. The total heat transmission rates, Q(t), at the surface of the fuel element are normalized by dividing by their initial values, Q(0), in each case. Figure 7 shows that the heat transmission rate in the case of a = 0.336 inch decreases more rapidly than that for the case of a = 0.500 inch. This result is obtained for $\kappa > 0$ and $\kappa = 0$. It is also seen that for both a = 0.500 inch and a = 0.336 inch the curve of Q(t) is smoother for $\kappa = 0$ than when $\kappa > 0$.

SUMMARY AND CONCLUSIONS

In heat transfer problems of nuclear engineering, Laplace transformation procedures have been used by various references (6). A method of Hankel transformations has been applied to develop the heat conduction equation 4 in this work with the advantage over the use of the Laplace transform that it is simpler to use. For example, it does not require the use of the complicated contour integration for the reactor heat transfer problems.

The integral, $\int_{t=0}^{t=1} n(t)\, e^{\alpha\lambda_m^2 t}\, dt$, by using the trapezoidal rule, is programmed on the IBM-360 digital computer with given values of α and λ_m. The quantity, n(t), as defined earlier, is not readily integrable in closed form The trapezoidal rule (11) has been employed to integrate the function n(t) in the computer program. Calculated values of n(t) as shown in Table 3 decrease rapidly from the start of sudden power reduction for 2 seconds, then decrease steadily at a lower rate.

The mathematical method which has been used in deriving equations for temperature distributions is an analytical technique, which provides particular solutions to heat conduction problems in nuclear power reactors.

The derived equations are applicable for calculations of the transient temperature distribution and surface heat transmission rate distribution in long solid cylindrical fuel elements for nuclear power reactors, in which the rate of heat generation varies with time in any prescribed manner. However, because of the material concepts involved it is impossible to apply the derived equations to cases of other geometries.

Equation 42 is applicable to problems in which boundary conductance and coolant temperature changed initially and heat generation rate varies in any prescribed manner from the steady-state value.

If the properties of the material remain constant and axial conductance is ignored, equation 44 may be used for problems in which heat generation rate changes with time and is also applicable in analyzing the mechanical stability of long solid cylindrical fuel elements of boiling water reactors.

Considering the change of properties, equation 58 is applicable for problems in which coolant temperature changes with time from an initial steady-state value. In developing equation 58, it has been tacitly assumed that the inlet coolant temperature remains constant after the power reduction.

Nuclear superheaters, which are usually operating at high temperatures, are of interest with respect to the current demand for nuclear power. Considering the concepts involved, the stability of fuel element for nuclear superheaters must be examined before they are operated for extended periods of time.

Natural uranium has been chosen as the fuel element in the example. However, the same procedures may be employed for ceramic fuel elements without any change in the derived equation for predicting transient temperature distributions and heat transmission rate distributions.

RECOMMENDATIONS FOR FURTHER STUDY

The analysis has been conducted only for the inner annular element, i.e., in the superheating region of the double annular fuel element. Consideration of both regions of the double annular fuel element, is one of the recommended subjects for further study to obtain more accurate and useful predictions of temperature distributions in the fuel element in nuclear superheaters. This would require using a somewhat different mathematical procedure than the one developed in this work.

The material in which heat generation takes place was assumed to be in the form of a long solid cylinder. Longitudinal heat conduction was neglected. In practical cases, the longitudinal heat conduction is considerable and must be included with radial heat conduction. The derived equations in this work represent the temperatures as functions of radial positions and time. Consideration of both the longitudinal and the radial heat conduction at transient state will result in the temperatures being given as functions of longitudinal distances, radial positions, and time.

Temperatures after the start of the sudden power reduction or shut-down of the reactor were calculated in this work, in which the fuel element was supposed to be placed in a uniform thermal neutron flux field. One of the most common techniques of power reduction of the reactor is sudden insertion of control rods into the core. By means of control rod motion, the thermal neutron flux is perturbed. Mathematical analysis of the transient temperature distributions after the reactor is perturbed is one of the subjects recommended for further study.

The assumed temperatures at which material properties were evaluated in this work were different from the calculated temperatures. A further study would make use of more realistic assumptions.

To simplify the heat conduction equation, the thermal conductivity, k, was assumed constant. In practical cases, the thermal conductivity changes with temperatures. If one considers the thermal conductivity as a function of temperature and develops the heat conduction equation, better predictions of fuel temperature distributions in the nuclear fuel element will be obtained.

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NOMENCLATURE

A	Constant, defined by equations 45a and 45b
A_1, A_2	Constants
a	Outer radius of the fuel element
B	Constant
C	Constant
c	Specific heat of the fuel element
D	Thermal diffusion coefficient
$f(r)$	Function of r
$\bar{f}_o$	Finite Hankel transform of order zero of the function f
G	Mean energy released per fission
h	U/k, boundary conductance
I_o, I_1	Modified Bessel functions of the first kind of zero and first orders, respectively
i	Index for the time increment, i.e., 1, 2, 3, ...
J_o, J_1	Bessel functions of first kind of order zero and one, respectively
$\bar{j}_o$	Finite Hankel transform of order zero
K_o	Modified Bessel function of the first kind of order zero
k	Thermal conductivity
M	Defined by equations 46a and 46b
m	Index for λ, i.e., 1, 2, 3, ...
$m(r)$	Radial temperature distribution
$\bar{m}(r)$	Finite Hankel transform of m(r)

$n(t)$	Function by which the heat generation rate varies with time
p	Index for summation increment
$Q(t)$	Heat transmission rate per unit length (Btu/hr°ft) with time t
Q_o	Heat transmission rate per unit length at initial steady-state from the fuel element
q	Heat generation rate per unit volume, $q = q_o m(r) n(t)$
q_o	Initial steady-state heat generation rate per unit volume in the fuel element
R	Defined by equation 47
r	Variable radius
T	Temperature of the fuel element
$\bar{T}$	Finite Hankel transform of T
T_o	Initial steady-state coolant temperature
T'	dT/dr
T''	d^2T/dr^2
t	Time
U	Overall heat transfer coefficient from the surface of the fuel element to the coolant
$x, y, z,$	Coordinates of the rectangular system
α	Thermal diffusivity, $\alpha = k/c\rho$
Δ	Finite difference, $\Delta t_i = t_i - t_{i-1}$
ϵ	Temperature difference, defined by equation 48
κ	Inverse thermal neutron diffusion length
λ	Parameter, defined by equation 8a

ρ Density of the fuel material

Σ Summation

Σ_f Macroscopic fission cross section

ϕ Thermal neutron flux

∇ The operator 'del'

∇^2 Laplacian operator

d/dx, d/dr Total derivatives with respect to x and r, respectively

$\partial/\partial x$, $\partial/\partial r$ Partial derivatives with respect to x and r, respectively

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